The Bloch Equations

The Bloch equations show up frequently in imaging, particularly for predicting the image contrast that will result from a particular imaging sequence, or for calculating how much steady state signal will be available. The goal of this discussion is to show the basic concepts involved in the derivation, and develop them through the point of describing relaxation in the absence of an RF pulse. We will leave the equations without discussion of the inclusion of B_1 (RF pulse) effects, which need to be considered to accurately model excitation of a sample, but can be ignored for a basic understanding of the imaging process.

The Bloch equations are a set of coupled differential equations that describe the behavior of the net magnetization in a sample under the influence of B_0. The following description is taken out of Chapter 4 in Haacke *et al.*, *Magnetic Resonance Imaging: Physical Principles and Sequence Design*. Wiley and Sons: New York (1999) ("the green book" at the CMRR).

Start with a single spin in a static magnetic field: how do you figure out what it's doing? There is no net force on the dipole, so we're concerned with the effect of torque on the angular momentum

$$\frac{d\vec{L}}{dt} = \sum \tau = \vec{\mu} \times \vec{B}$$

Remember the relationship between angular momentum, L, and magnetic moment, μ, for a spin; it is determined by the gyromagnetic ratio ($\mu = \gamma L$), so the equation becomes:

$$\frac{d\vec{\mu}}{dt} = \gamma \vec{\mu} \times \vec{B}$$

For a population of spins, the net magnetic moment is given by:

$$\vec{M} = \sum_i \vec{\mu}_i$$

Resulting in:

$$\frac{d\vec{M}}{dt} = \gamma \vec{M} \times \vec{B}$$

The net magnetization is a vector, as is the field, **B**. For simplicity, we will consider a uniform magnetic field **B** oriented in the z direction: **B** = $B_0 \hat{z}$. In the absence of interactions between spins (no relaxation) the three equations generated by the cross product are:

$$\frac{dM_x}{dt} = \gamma M_y B_0 \hat{x} = \omega_0 M_y$$
$$\frac{dM_y}{dt} = -\gamma M_x B_0 \hat{y} = -\omega_0 M_x$$
$$\frac{dM_z}{dt} = 0$$

This steady-state solution, without relaxation, shows how precession is determined by the cross-product, since the solution to these coupled differential equations will be

$$M_x(t) = M_x(0)\cos\omega_0 t + M_y(0)\sin\omega_0 t$$
$$M_y(t) = M_y(0)\cos\omega_0 t - M_x(0)\sin\omega_0 t$$

There are several different ways of representing the transverse magnetization, depending on the application ($M_+ = M_x + iM_y$, $M_\perp = M_x\hat{x} + M_y\hat{y}$).

To add in a relaxation term, we will start with the longitudinal magnetization, M_z. We know that once perturbed, this component of the magnetization will recover at a rate proportional to the difference between the steady-state magnetization, M_0, and the current magnetization:

$$\frac{dM_z}{dt} = \frac{1}{T_1}(M_0 - M_z)$$

The solution to this "first order differential equation with an inhomogeneity" is

$$M_z(t) = M_z(0)e^{-t/T_1} + M_0(1 - e^{-t/T_1})$$

The steady-state solution can be verified by looking at the answer for $t \to \infty$.

For the transverse magnetization, adding decay at a rate $1/T_2$ gives us the following differential equations to solve:

$$\frac{dM_x}{dt} = \omega_0 M_y - \frac{1}{T_2}M_x$$
$$\frac{dM_y}{dt} = -\omega_0 M_x - \frac{1}{T_2}M_y$$

Resulting in:

$$M_x(t) = e^{-t/T_2}(M_x(0)\cos\omega t + M_y(0)\sin\omega t)$$
$$M_y(t) = e^{-t/T_2}(M_y(0)\cos\omega t - M_x(0)\sin\omega t)$$

Which can be more simply represented as $M_\perp = M_\perp(0)e^{-t/T2}$ in the rotating frame.

These equations have been derived assuming always a constant $\mathbf{B} = B_0\hat{z}$. Therefore, these solutions describe only relaxation after a pulse, or after perturbation from equilibrium. The pulse can be accounted for by realizing that, during the RF pulse, $\mathbf{B}_{eff} = B_{1x}\hat{x} + B_{1y}\hat{y} + B_0\hat{z}$.

The Bloch Equations: predicting image contrast and calculating the Ernst angle

So now we have the Bloch equations with the form:

$$M_z(t) = M_z(0)e^{-t/T_1} + M_0(1 - e^{-t/T_1})$$
$$M_\perp(t) = M_\perp(0)e^{-t/T_2}$$

A brief reminder: these equations came from solving the basic equation:

$$\frac{d\overrightarrow{M}}{dt} = \gamma \overrightarrow{M} \times \vec{B}$$

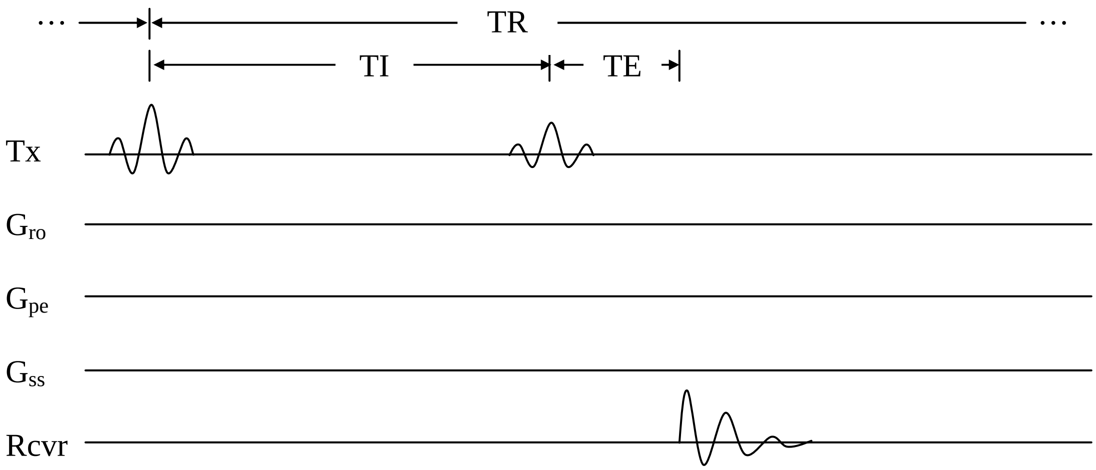
for the description of steady-state precession, then adding decay terms to describe observed behavior. These equations describe *only* the relaxation of magnetization in the absence of RF pulses. The steady-state solution can be verified by looking at the answer for $t \to \infty$.

Without adding the complication of solving the Bloch equations during the pulse, we can use these equations to understand image contrast – how much signal is found in what kind of tissue at the time the signal is acquired. For this discussion, we'll define the following terms:

- TR: repetition time of an experiment (time between repeated RF pulses)
- TE: echo time, time between excitation pulse and acquisition of center of k-space
- TI: inversion time, time between 180 inversion pulse and excitation pulse in a magnetization-prepared experiment
- T_1: longitudinal (spin-latice) relaxation rate
- T_2: transverse (spin-spin) relaxation rate
- M_0: equilibrium magnetization
- M_{ss}: steady-state magnetization

The process of using the Bloch equations to predict the outcome of an experiment will be a combination of intuition and mathematics. We'll use the Bloch equations to describe how magnetization decays after a pulse, and we'll manually insert the effect of the RF pulse. To the best of my knowledge, analytical solution of the Bloch equations during the pulse is rarely possible, and programs such as PulseTool apply a numerical solution, based on the sum of a series of low-angle flips.

We’ll start with a simple inversion recovery experiment (no imaging gradients):



* Not drawn to scale – TE is much shorter than TI.

Now we’ll just walk through the sequence one step at a time, predicting the longitudinal and transverse magnetization at each step.

IF TR >> T_1 (or if this experiment is only performed once), then the magnetization just before the application of the first pulse is fully relaxed:

$$M_z(0^-) = M_0$$
$$M_\perp(0^-) = 0$$

After the inversion pulse, the longitudinal magnetization is flipped 180°, and the transverse magnetization is still zero:

$$M_z(0^+) = -M_0$$
$$M_\perp(0^+) = 0$$

Then, during the delay before the next pulse, the longitudinal magnetization decays. Just before the excitation pulse, the magnetization (at time t = TI^-) is:

$$M_z(TI^-) = M_z(0^+)e^{-TI/T_1} + M_0(1-e^{-TI/T_1}) = -M_0 e^{-TI/T_1} + M_0(1-e^{-TI/T_1}) = M_0(1-2e^{-TI/T_1})$$
$$M_\perp(TI^-) = 0$$

We will let the excitation pulse be any flip angle, not just 90°, and for a minute, we will not assume that the transverse magnetization is 0:

$$M_z(TI^+) = M_z(TI^-)\cos(\alpha) + M_\perp(TI^-)\sin(\alpha)$$
$$M_\perp(TI^+) = M_\perp(TI^-)\cos(\alpha) + M_z(TI^-)\sin(\alpha)$$

This equation points out the complication that arises if transverse magnetization does not decay or get crushed between experiments – stimulated echoes are generated, which makes the evolution of the magnetization much more difficult to follow (although sometimes these stimulated echoes are actually very useful). For now, we'll assume that TR >> T_2, so $M_\perp$ is still 0:

$$M_z(TI^+) = M_z(TI^-)\cos(\alpha) = M_0(1-2e^{-TI/T_1})\cos(\alpha)$$
$$M_\perp(TI^+) = M_z(TI^-)\sin(\alpha) = M_0(1-2e^{-TI/T_1})\sin(\alpha)$$

Finally, we calculate the magnetization when the data is acquired, at t=TE:

$$M_z(TE) = M_z(TI^+)e^{-TE/T_1} + M_0(1-e^{-TE/T_1}) = M_0(1-2e^{-TI/T_1})\cos(\alpha) + M_0(1-e^{-TE/T_1})$$
$$M_\perp(TE) = M_\perp(TI^+)e^{-TE/T_2} = M_0(1-2e^{-TI/T_1})\sin(\alpha)e^{-TE/T_2}$$

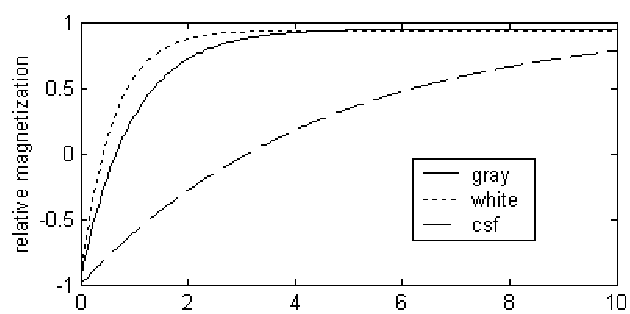
The longitudinal magnetization is invisible to our RF coil, so our signal is related just to the transverse magnetization. Now we have an equation that describes our signal in an inversion-recovery experiment (which starts out with a fully relaxed sample, so TR is not part of the equation) as a function of TI, T_1, α, TE, and T_2.

<u>Exercises</u>

1) For an anatomical image acquired at 1.5T, where the T_1 and T_2 values for grey matter, white matter, and CSF are given in the table below, predict the *relative signal* strength (i.e., if $M_{0,GM}=M_{0,WM}=M_{0,CSF}$) in the three compartments when an image is acquired with flip angle α = 90°, TI = 3.1s, and TE = 5ms.
2) How would the answer be different if α = 45°?

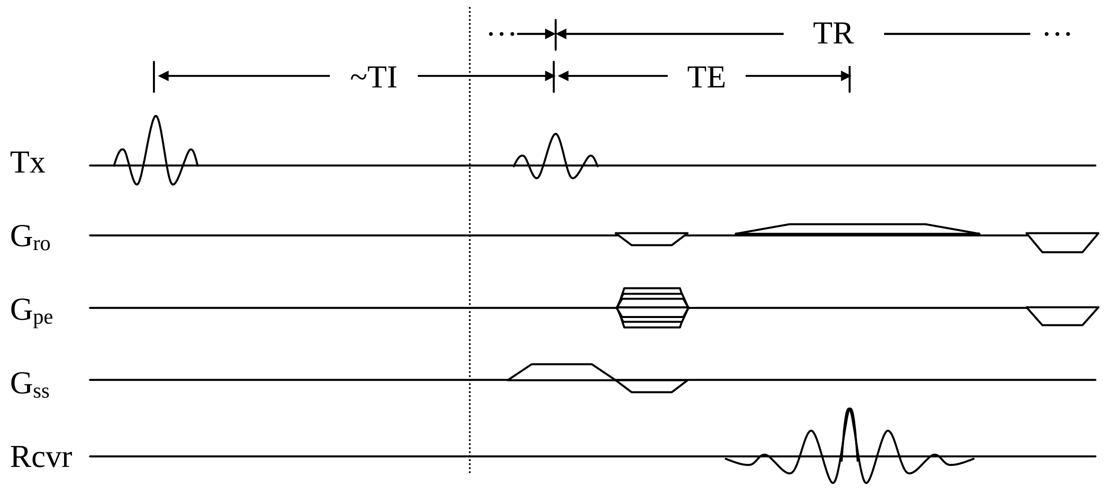
Tissue relaxation times at 1.5T taken from Haacke, Ch 1., pg. 9:

Tissue	T_1 (ms)	T_2 (ms)
Grey matter	950	100
White matter	600	80
Cerebrospinal fluid	4500	2200



Imaging with TR << T_1

A simple imaging experiment:



*Note that we've pulled the inversion pulse (the magnetization preparation) outside of the imaging loop. If we can get the entire image acquired on a time frame that is small relative to the T_1's, then the T_1 contrast is the same throughout the image and we only needed one 180° pulse.

If the excitation flip angle is large, then a large proportion of the longitudinal magnetization is put down into the transverse plane for the first read-out line, then crushed after that line (those gradients on the end of G_{ro} and G_{pe} lines are crusher gradients, designed to dephase any transverse magnetization that remains after the read-out, in order to avoid having it come back into the image on the next repetition). If TR << T1, and the excitation flip angle is large, then only a small fraction of the longitudinal magnetization recovers before everything is crushed (spoiled) in the transverse plane (e.g. only $1-e^{-.010/.95} = .01$ remains on the longitudinal axis), so for the next iteration, only a tiny fraction of the magnetization remains.

Solution: use a low flip angle, so only a little of the magnetization is spent on each repetition.

Calculating the Ernst angle for repeated measurements

Functional imaging experiments have a similar problem. Scans are repeated with repetition times >> T_2, so we don't have to worry about transverse magnetization and crusher gradients. But we still have to worry about the fact that TR is often less than T_1. When this is the case, it is ideal to use a flip angle less than 90°, for the same reason that anatomical imaging sequences use a low flip angle. The flip angle that maximizes available steady state magnetization in either a functional or an anatomical scan is called the Ernst angle. Here's the derivation:

We will pick up the experiment in the middle, when we've reached steady state, using the expression $M_z(t_n)$ to denote the longitudinal magnetization after the n^{th} pulse (n^{th} TR):

$$M_z(t_{n+1}) = M_z(t_n)\cos(\alpha)e^{-TR/T_1} + M_0(1-e^{-TR/T_1})$$

The first part of that expression describes how much magnetization was left after the pulse, and the second part describes how much recovered during the TR. In the steady-state, $M_z(t_n) = M_z(t_{n+1}) = M_{ss}$, producing:

$$M_{ss} = \frac{M_0(1-e^{-TR/T_1})}{1-\cos(\alpha)e^{-TR/T_1}}$$

The magnetization in the transverse plane is then:

$$M_\perp(TR^+) = M_{ss}\sin(\alpha) = \frac{M_0(1-e^{-TR/T_1})}{1-\cos(\alpha)e^{-TR/T_1}}\sin(\alpha)$$

So we simply want to find the flip angle, α, that maximizes the magnitude in the above equation. Taking the derivative with respect to α and setting it equal to zero produces:

$$\cos(\alpha) = e^{-TR/T_1}$$

So the Ernst angle, α, is

$$\alpha = \cos^{-1}(e^{-TR/T_1})$$